Effective temperature in nonequilibrium steady states of Langevin systems with a tilted periodic potential

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We theoretically study Langevin systems with a tilted periodic potential. It is known that the ratio Θ of the diffusion constant D to the differential mobility $\mu_{\rm d}$ is not equal to the temperature of the environment (multiplied by the Boltzmann constant), except in the linear response regime, where the fluctuation dissipation theorem holds. In order to elucidate the physical meaning of Θ far from equilibrium, we analyze a modulated system with a slowly varying potential. We derive a large scale description of the probability density for the modulated system by use of a perturbation method. The expressions we obtain show that Θ plays the role of the temperature in the large scale description of the system and that Θ can be determined directly in experiments, without measurements of the diffusion constant and the differential mobility. Hence the relation $D=\mu_{\rm d}\Theta$ among the independent measurable quantities D, $\mu_{\rm d}$, and Θ can be interpreted as an extension of the Einstein relation.

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I. INTRODUCTION

Technological development in both the manipulation and observation of objects on a small scales has led to further understanding of the behavior exhibited by mechanical systems of length scales $\sim 10^{-6}$ m, force scales $\sim 10^{-12}$ N, and time scales $\sim 10^{-3}$ sec. One such system that has been studied consists of a small bead suspended in a fluid. Because of their simplicity, such systems are particularly useful for studying topics relevant to fundamental physics. Indeed, recently, the fluctuation theorem [1] and the Jarzynski equality [2], which were derived theoretically as relations universally valid for nonequilibrium processes, have been verified experimentally through experiments on systems consisting of small beads [3] and RNA molecules [4] employing optical tweezers.

In studying nonequilibrium systems, we want to discover the uniquely nonequilibrium behavior as well as to determine what properties of equilibrium systems remain even far from equilibrium. We believe that small systems are suited for such studies because nonequilibrium effects become more significant as the system size decreases. In particular, with regard to nonequilibrium systems, we are interested in finding relations between measurable quantities that may be useful in the construction of a systematic theory of nonequilibrium statistical mechanics.

In the present paper, we study the motion of a small bead suspended in a fluid of temperature T. The bead is confined to move in a single direction, say the x direction, and is subjected to a periodic potential $U(x)$ of period ℓ. Such a system can be realized experimentally as a scanning optical trap system [6], for example. Further, a flow with constant velocity can be used to apply a constant driving force f to the bead. In this way, it is possible to experimentally realize nonequilibrium steady states (NESSs) for such a bead system.

The quantity we investigate is the ratio Θ of the diffusion constant D to the differential mobility $\mu_{\rm d}$ for a bead in a NESS. In the linear response regime, the ratio Θ is identical to the temperature of the environment (multiplied by the Boltzmann constant). This relation is equivalent to one form of the fluctuation dissipation theorem (FDT). However, because the FDT does not hold for a NESS far from equilibrium, Θ is not identical to the temperature of the environment. Nevertheless, in the system considered there, we find that Θ plays the role of the temperature in the description of the large scale behavior of the system and that Θ can be determined experimentally in a direct manner, without the need to measure D and $\mu_{\rm d}$. We obtain this result by employing a perturbation method to derive the large scale description of the probability density.

II. MODEL

We assume that the motion of the bead is described by the one-dimensional Langevin equation

$$\gamma\dot{x} = -\frac{\partial U(x)}{\partial x} + f + \sqrt{2\gamma T}\xi(t), \tag{1}$$

where $\xi(t)$ is a Gaussian white noise with zero mean and unit dispersion. The Boltzmann constant is set to unity. We note that there are many physical examples that are described by this equation [6]. Here, we make two remarks on the form of Eq. (1). First, inertial effects are considered to be negligible, because a typical value of the relaxation time of the particle velocity, which is estimated to be $\sim 10^{-9}$ sec [7], is much shorter than the characteristic time scale of the phenomena observed in the type of experiments we consider. Second, T in Eq. (1) is assumed to be the temperature of the environment. Although the validity of this assumption is not known for NESSs in general, our result does not depend on the

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physical interpretation of T, because we do not need to use the value of T in our analysis.

The probability density for the position of the particle $p(x,t)$ in this system obeys the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \frac{1}{\gamma}\frac{\partial}{\partial x}\left[\left(\frac{\partial U}{\partial x} - f\right)p + T\frac{\partial p}{\partial x}\right]. \tag{2}$$

The steady state density $p_s(x;f)$ is obtained as [8]

$$p_s(x;f) = \frac{1}{Z}I_-(x) \tag{3}$$

with

$$I_-(x) = \int_0^\ell dy e^{-\beta U(x)+\beta U(y+x)-\beta f y}, \tag{4}$$

where $\beta = 1/T$ and Z is a normalization factor by which

$$\int_0^\ell dx p_s(x;f) = \ell. \tag{5}$$

The steady state current $v_s(f)$ is derived as

$$v_s(f) = \frac{T}{\gamma}\frac{1 - e^{-\beta f \ell}}{(1/\ell)\int_0^\ell dx I_-(x)}. \tag{6}$$

The exact expression for the diffusion constant $D(f)$ has been derived recently for the system under consideration [9]. By using a different method (see Secs. IV and VI, we derive the following form for $D(f)$:

$$D(f) = \frac{T}{\gamma}\frac{(1/\ell)\int_0^\ell dx[I_-(x)]^2 I_+(x)}{[(1/\ell)\int_0^\ell dx I_-(x)]^3}, \tag{7}$$

where

$$I_+(x) = \int_0^\ell dy e^{\beta U(x)-\beta U(x-y)-\beta f y}. \tag{8}$$

Note that Eq. (7) is equivalent to the expression derived in Ref. [9], which is obtained by simply exchanging I_+ and I_- in Eq. (7).

In the inset of Fig. 1, we display an example of $D(f)$ for the case $U(x) = U_0 \sin 2\pi x/\ell$. It can be seen that the diffusion is enhanced around $f\ell/T = 2\pi U_0/T$. This effect was first reported in Ref. [10] and was subsequently analyzed more quantitatively [9].

III. QUESTION

In the linear response regime, the mobility μ, defined as $\mu = \lim_{f\to 0} v_s(f)/f$, is equal to $D(f=0)T$. This is one form of the FDT. However, for a NESS far from equilibrium, the FDT does not hold. In fact, far from equilibrium, there does not even exist a FDT involving the differential mobility $\mu_d \equiv dv_s/df$, drived from Eq. (6) as follows:

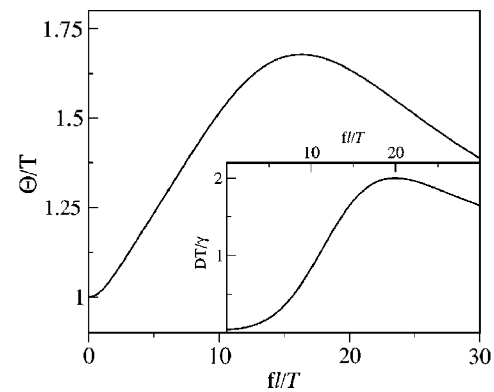

FIG. 1. Θ/T as a function of $f\ell/T$ in the case that $U(x) = U_0 \sin 2\pi x/\ell$ with $U_0/T = 3.0$. $\Theta/T = 1$ at $f=0$ and $\Theta/T > 1$ for $f > 0$. The inset displays $D\gamma/T$ as a function of $f\ell/T$ calculated using Eq. (7).

$$\mu_d(f) = \frac{1}{\gamma}\frac{(1/\ell)\int_0^\ell dx I_-(x)I_+(x)}{[(1/\ell)\int_0^\ell dx I_-(x)]^2}. \tag{9}$$

In order to determine quantitatively the extent to which the FDT is violated, we define Θ as the ratio of D to μ_d:

$$\Theta(f) \equiv \frac{D(f)}{\mu_d(f)}. \tag{10}$$

As displayed in Fig. 1, although the dimensionless quantity Θ/T is unity in equilibrium, it depends on $f\ell/T$ and on the form of the periodic potential $U(x)$ when the system is far from equilibrium. In addition to the fact that the FDT does not hold far from equilibrium, no means of measuring Θ experimentally without measuring D and μ_d has been proposed. Hence, there is no known method of deducing μ_d from experimentally measured values of D. While the fluctuation theorem reported in Ref. [1], which is valid for a NESS far from equilibrium, can be regarded as an extension of FDT [11], it seems difficult to connect it to a physical interpretation of Eq. (10).

Despite the apparent difficulties described above, we attempt to propose a method of measuring Θ experimentally without measuring D and μ_d. We first note that Θ/T ($\neq 1$ in a NESS) is interpreted as the FDT violation factor. Recently, stimulated by a proposal for the thermodynamic measurement of the FDT violation factor in spin glass systems as the "effective temperature" [12], the feasibility of physical measurements of the effective temperature has been investigated in numerical experiments modeling a sheared glassy material [13], a driving system near jamming [14], and driven vortex lattices [15]. Among these, Berthier and Barrat proposed a measurement method of the FDT violation factor as the effective temperature in sheared glassy systems [13]. They used a tracer particle of large mass as an effective thermometer and demonstrated that the kinetic energy of the tracer is related to the FDT violation factor. This result exhibits a clear relation between the FDT violation factor and the measurable effective temperature.

Here too we seek to determine whether there is a realm in which Θ can be interpreted as the effective temperature in a NESS and, hence, whether it can be measured independently of D and μ_d.

IV. RESULTS

We propose a method of measuring Θ by adding a slowly varying potential which plays the role of an effective thermometer to the system under consideration. In order to demonstrate this method, we study Eq. (1) with $U(x)$ replaced by a potential $U(x)+V(x)$, where $V(x)$ is a slowly varying periodic potential $V(x)$ of period $L \gg \ell$. We investigate the behavior of the probability density on length scales larger than L. In order to make the separation of scales explicit, we define $\varepsilon \equiv \ell / L$ and the large scaled coordinate $X \equiv \varepsilon x$. The probability density in this system obeys the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \frac{1}{\gamma}\frac{\partial}{\partial x}\left[\left(\frac{\partial U}{\partial x} + \varepsilon\frac{\partial \tilde{V}}{\partial X} - f\right)p + T\frac{\partial p}{\partial x}\right], \tag{11}$$

where we have defined $\tilde{V}(X) \equiv V(x)$.

We extract the large scale behavior of $p(x,t)$ by introducing a slowly varying field $Q(X,t)$ as

$$p(x,t) = p_s(x;f_m)\left(Q + \varepsilon a(x;f_m)\frac{\partial Q}{\partial X} + \varepsilon^2 b(x;f_m)\frac{\partial^2 Q}{\partial X^2} + \varepsilon c(x;f_m)\frac{\partial f_m}{\partial X}Q + O(\varepsilon^3)\right), \tag{12}$$

where $f_m = f - \varepsilon\,\partial\tilde{V}/\partial X$. Because f_m is a function of $X = \varepsilon x$, the x dependence of $p_s(x;f_m)$ appears in two ways, as an explicit dependence and as a dependence through f_m. Note that $p_s(x;f_m)$ is a periodic function in the sense that $p_s(x+\ell;f_m) = p_s(x;f_m)$. The functions $a(x;f_m)$, $b(x;f_m)$, and $c(x;f_m)$ are similar; that is, they are periodic functions of x in the same sense and depend on f_m. Their functional forms are determined below.

Substituting Eq. (12) into Eq. (11), we obtain

$$\frac{\partial Q}{\partial t} = \varepsilon\mathcal{A}\frac{\partial Q}{\partial X} + \varepsilon^2\mathcal{B}\frac{\partial^2 Q}{\partial X^2} + \varepsilon\mathcal{C}\frac{\partial f_m}{\partial X}Q + O(\varepsilon^3), \tag{13}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ can be expressed by

$$\mathcal{A} = -\left(v_s - \frac{T}{\gamma}p_s'\right)p_s^{-1}(1+a') + \frac{T}{\gamma}a'', \tag{14}$$

$$\mathcal{B} = -\left(v_s - \frac{T}{\gamma}p_s'\right)p_s^{-1}(a+b') + \frac{T}{\gamma}(1+2a'+b'') - a\mathcal{A}, \tag{15}$$

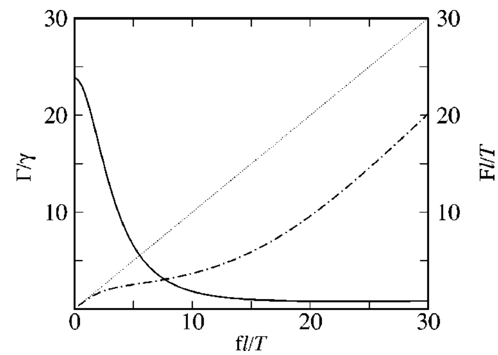

FIG. 2. The left axis represents Γ/γ, which is displayed as a function of $f\ell/T$ (solid curve), and the right axis represents $F\ell/T$, also displayed as a function of $f\ell/T$ (dash-dotted curve), in the case that $U(x) = U_0 \sin 2\pi x/\ell$, with $U_0/T = 3.0$. The dotted line corresponds to $F = f$.

$$\mathcal{C} = -\left(v_s - \frac{T}{\gamma}p_s'\right)p_s^{-1}c' + \frac{T}{\gamma}c'' - \frac{dv_s}{df_m}p_s^{-1} + \frac{T}{\gamma}\frac{\partial p_s'}{\partial f_m}p_s^{-1}, \tag{16}$$

where the prime represents the partial derivative with respect to x, that is, $a'(x;f_m) = \partial a(x;f_m)/\partial x$ and so on. Then we can choose a, b, and c so that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ do not depend on x explicitly but depend on x through the X dependence of f_m. After a straightforward calculation, we find that, subject to this condition, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are uniquely determined as

$$\mathcal{A} = -v_s(f_m), \tag{17}$$

$$\mathcal{B} = D(f_m), \tag{18}$$

$$\mathcal{C} = -\frac{dv_s(f_m)}{df_m}. \tag{19}$$

(The derivation will be presented in Sec. VI) Using this result, we rewrite Eq. (13) as

$$\frac{\partial Q}{\partial t} = \varepsilon\frac{\partial}{\partial X}\left[-v_s(f_m)Q + D(f_m)\varepsilon\frac{\partial Q}{\partial X} + O(\varepsilon^2)\right]. \tag{20}$$

Recalling that $f_m = f - \varepsilon\,\partial\tilde{V}/\partial X$, we see that this equation is equivalent to the Fokker-Planck equation

$$\frac{\partial Q}{\partial t} = \frac{1}{\Gamma}\varepsilon\frac{\partial}{\partial X}\left[\left(\varepsilon\frac{\partial\tilde{V}}{\partial X} - F\right)Q + \Theta\varepsilon\frac{\partial Q}{\partial X} + O(\varepsilon^2)\right], \tag{21}$$

where we have defined

$$\Gamma(f) \equiv \mu_d(f)^{-1}, \tag{22}$$

$$F(f) \equiv v_s(f)\mu_d(f)^{-1}. \tag{23}$$

For reference, in Fig. 2, we present graphs of Γ and F as a function of f for the model considered in Fig. 1.

We note that Eq. (21) has the same form as Eq. (2), with the parameters Γ, F, and Θ corresponding to γ, f, and T, respectively. Thus, replacing $(\gamma, f, T, U(x))$ in Eq. (6) by

$(\Gamma, F, \Theta, \tilde{V}(X))$, we obtain the current for a NESS of this modulated system. From this result, by measuring the steady state current for several forms of $V(x)$, we can determine the values of Γ, F, and Θ experimentally, where we note that these values do not depend on the choice of $V(x)$. This fact implies the existence of the relation $D=\mu_d\Theta$ among the independent measurable quantities D, μ_d, and Θ in a NESS far from equilibrium. This relation $D=\mu_d\Theta$ can be thought of as an extended Einstein relation. Furthermore, from the correspondence between Θ in Eq. (21) and T in Eq. (2), it is evident that Θ plays the role of the temperature for the large scale behavior of the system. In this way, we have arrived at the main claim of this paper, the physical interpretation of Θ.

Here we address two remarks on the main claim. First, one may naively expect that on the large scale Eq. (1) can be effectively described by

$$\dot{x} = v_s(f) + \sqrt{2D(f)}\xi(t). \tag{24}$$

Although such an effective description is valid, we emphasize that the effective temperature is not determined from this description. In order to have the correspondence with Eq. (21), we need one more quantity in addition to D and v_s. In our analysis, by adding the slowly varying potential to the system, Γ, F, and Θ are determined. We thus interpret the slowly varying potential as an effective thermometer for the original system described by Eq. (1).

Second, we have another method of measuring Θ by considering the case in which a slowly varying potential $\tilde{V}(\varepsilon(x-v_s t))$ that moves with constant velocity v_s replaces $\tilde{V}(\varepsilon x)$ in Eq. (11). Although the resulting system is more complicated than that considered above, the large scale behavior in this case is actually simpler. Indeed, using the same method as above, we obtain the following equation for $Q(Y,t)$ describing the large scale behavior of the system:

$$\frac{\partial Q(Y,t)}{\partial t} = \frac{1}{\Gamma}\varepsilon\frac{\partial}{\partial Y}\left(\varepsilon\frac{\partial \tilde{V}(Y)}{\partial Y}Q + \Theta\varepsilon\frac{\partial Q}{\partial Y} + O(\varepsilon^2)\right). \tag{25}$$

Here, we have introduced the large scaled moving coordinate $Y \equiv \varepsilon(x-v_s t)$. This equation is identical to the Fokker-Planck equation describing the time evolution of the probability density in an equilibrium state with temperature Θ. Therefore, for example, Θ is obtained by measuring the statistical average of $\tilde{V}(Y)$.

V. NUMERICAL EXPERIMENT

Let us demonstrate that the measurement method of Θ by use of a slowly varying potential works well in numerical experiments. We study Eq. (1) numerically with periodic boundary conditions for the case that $U(x)=U_0 \sin 2\pi x/\ell$ and that the system size is L. The parameter values are chosen as follows: $L/\ell=50$, $U_0/T=3$, and $f\ell/T=16$.

We measure the effective temperature for this system by using a moving potential $V(y)$ in the form

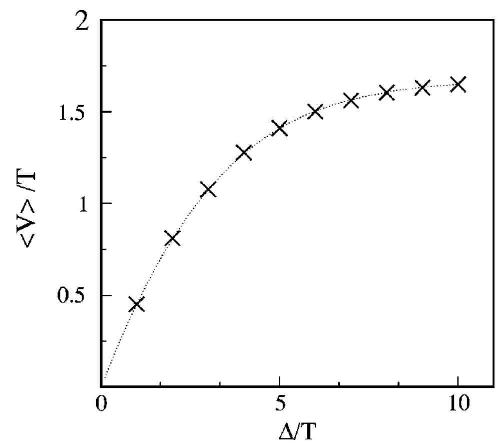

FIG. 3. $\langle V\rangle/T$ versus Δ/T obtained in the numerical experiment. The dotted line represents a fitting curve of the form (27) in which $\Theta/T=1.67(4)$.

$$V(y) = \begin{cases} 2\Delta y/L & \text{for} \quad 0 \leq y \leq L/2, \\ 2\Delta - 2\Delta y/L & \text{for} \quad L/2 \leq y \leq L, \end{cases} \tag{26}$$

with $y=\mathrm{mod}(x-v_s t, L)$. The statistical averages of V in a NESS, $\langle V\rangle$, are measured for several values of Δ. As displayed in Fig. 3, the graph $(\Delta, \langle V\rangle)$ is fitted well by the form

$$\langle V\rangle = \Theta - \frac{\Delta}{\exp(\Delta/\Theta) - 1}, \tag{27}$$

which might be expected from Eq. (25). Using this fitting for the experimental result, we evaluate the value of Θ/T as $\Theta/T=1.67(4)$. This value should be compared with the theoretical one, 1.677, which is calculated from Eq. (10). Thus we conclude that our measurement method for Θ works well in experiments.

VI. TECHNICAL DETAILS

We regard Eq. (14) as an ordinary differential equation (ODE) for $a(x;f_m)$ under the condition that $\mathcal{A}$ does not depend on x. This ODE has periodic solutions for x only when $\mathcal{A}$ satisfies a certain condition, which provides $\mathcal{A}$ uniquely.

Let us solve the ODE (14). We first define

$$d(x;f_m) = a'(x;f_m)p_s(x;f_m). \tag{28}$$

Then Eq. (14) becomes

$$\frac{T}{\gamma}d' - v_s a' - v_s + \frac{T}{\gamma}p_s' - \mathcal{A}p_s = 0. \tag{29}$$

Integrating this equation over the range $[0,\ell]$, we obtain

$$\mathcal{A} = -v_s, \tag{30}$$

where we have used a requirement that $Td/\gamma - v_s a$ is a periodic function.

Under the condition (30), we can derive periodic solutions $a(x;f_m)$. The integration of Eq. (29) leads to

$$\frac{T}{\gamma}a' p_s - v_s a - v_s x + \frac{T}{\gamma}p_s + v_s H + v_s K_1 = 0, \tag{31}$$

where K_1 is a constant and we have defined

$$H(x;f_m) = \int_0^x dy p_s(y;f_m). \tag{32}$$

Substituting the expression

$$a(x;f_m) = H(x;f_m) - x + \overline{a}(x;f_m) \tag{33}$$

into Eq. (31), we rewrite Eq. (31) as

$$\frac{T}{\gamma}p_s\overline{a}' - v_s\overline{a} + \frac{T}{\gamma}p_s^2 + v_s K_1 = 0. \tag{34}$$

Noting the relation

$$v_s(p_s e^{\beta[U(x)-f_m x]}) = -\frac{T}{\gamma}p_s(p_s e^{\beta[U(x)-f_m x]})', \tag{35}$$

we solve Eq. (34) as

$$\overline{a}(x;f_m) = -p_s(x;f_m)^{-1}e^{-\beta[U(x)-f_m x]}\Phi(x;f_m) + K_2 p_s(x;f_m)^{-1}e^{-\beta[U(x)-f_m x]} + K_1, \tag{36}$$

where we have defined

$$\Phi(x;f_m) = \int_0^x dy p_s(y;f_m)^2 e^{\beta[U(y;f_m)-f_m y]}. \tag{37}$$

From the condition $a(0;f_m)=a(\ell;f_m)$, the constant K_2 is determined as

$$K_2 = \frac{1}{1-e^{-\beta f_m \ell}}\Phi(\ell;f_m). \tag{38}$$

Equations (33) and (36)–(38) provide all periodic solutions of the ODE (14).

Next we study Eq. (15) by repeating a similar analysis. Defining

$$h(x;f_m) = b'(x;f_m)p_s(x;f_m), \tag{39}$$

we rewrite Eq. (15) as

$$\frac{T}{\gamma}h' - v_s b' - v_s a + \frac{T}{\gamma}(1+2a')p_s + \frac{T}{\gamma}a p_s' - \mathcal{B}p_s + a v_s p_s = 0. \tag{40}$$

Integrating this equation over the range $[0,\ell]$, we obtain

$$\mathcal{B} = \frac{T}{\gamma} + \frac{T}{\gamma\ell}\int_0^\ell dx a' p_s - \frac{v_s}{\ell}\int_0^\ell dx a(1-p_s), \tag{41}$$

where we have used the requirement that $Th/\gamma - v_s b$ is a periodic function of x. We can derive the expression for $\mathcal{B}$ by substituting $a(x;f_m)$ into Eq. (41). However, this expression is very complicated. We now simplify this.

Eliminating a' in Eq. (41) by use of Eqs. (33) and (34), we obtain

$$\mathcal{B} = \frac{v_s}{\ell}\int_0^\ell dx(x-H) + \frac{v_s}{\ell}\int_0^\ell dx p_s a - v_s K_1. \tag{42}$$

We can simplify this equation as

$$\mathcal{B} = \frac{v_s}{\ell}\int_0^\ell dx p_s\overline{a} - v_s K_1, \tag{43}$$

where we have used the equality

$$\int_0^\ell dx(x-H)(1-p_s) = 0. \tag{44}$$

Substituting Eq. (36) into Eq. (43), we obtain

$$\mathcal{B} = \frac{v_s}{\ell(1-e^{-\beta f_m \ell})}\int_0^\ell dx\, e^{-\beta[U(x)-f_m x]}[\Phi(\ell;f_m) - (1 - e^{-\beta f_m \ell})\Phi(x;f_m)]. \tag{45}$$

Here we have an identity for an arbitrary periodic function $\phi(x)$ with the period ℓ:

$$\int_0^\ell dy \phi(y)e^{-\beta f_m y} - (1-e^{-\beta f_m \ell})\int_0^x dy\phi(y)e^{-\beta f_m y} = e^{-\beta f_m x}\int_0^\ell dy\phi(y+x)e^{-\beta f_m y}. \tag{46}$$

Putting $\phi = p_s^2 e^{\beta U}$ in Eq. (46), we simplify Eq. (45) as

$$\mathcal{B} = \frac{v_s}{(1-e^{-\beta f_m \ell})\ell}\int_0^\ell dx e^{-\beta U(x)} \times \int_0^\ell dy[p_s(y + x;f_m)]^2 e^{\beta U(y+x) - \beta f_m y}. \tag{47}$$

Using Eqs. (3) and (6), we can derive

$$\mathcal{B} = D(f_m) \tag{48}$$

with Eq. (7).

Finally, defining

$$g(x;f_m) = c'(x;f_m)p_s(x;f_m), \tag{49}$$

we rewrite Eq. (16) as

$$\frac{T}{\gamma}g' - v_s c' - \frac{dv_s}{df_m} + \frac{T}{\gamma}\frac{\partial p_s'}{\partial f_m} - p_s\mathcal{C} = 0. \tag{50}$$

Integrating this equation over the region $[0,\ell]$, we obtain

$$\mathcal{C} = -\frac{dv_s}{df_m}, \tag{51}$$

where we have used the requirement that $Tg/\gamma - v_s c$ is a periodic function of x.

VII. DISCUSSION

In conclusion, we have proposed a method of measuring the effective temperature of the Langevin system (1) by using a slowly varying potential and have found that this effective temperature is equal to Θ defined by Eq. (10). The independence of measurements of the quantities D, μ_d, and Θ makes us interpret $D=\mu_d\Theta$ as the extended Einstein relation of the Langevin equation. This significant result was obtained by analyzing the Fokker-Planck equation with a slowly varying potential.

At the end of this paper, we shall present remarks on Eq. (25), which is derived by analysis of the system with a moving potential. Equation (25) provides two important insights in addition to the direct measurement method of Θ.

The first insight obtained from Eq. (25) is related to the interpretation of the extended Einstein relation $D=\mu_d\Theta$ among independent measurable quantities. Because Eq. (25) is identical in form to equations that describe equilibrium systems, $D=\Theta/\Gamma$ (the Einstein relation in the linear response theory) should hold. From Eq. (22), this Einstein relation yields $D=\mu_d\Theta$. As is well known, the Einstein relation is closely connected to the existence of detailed balance for fluctuations. However, note that fluctuations described by Eq. (1) with $f\neq 0$ do not satisfy the detailed balance condition, as can easily be checked by using the steady state density (3). Therefore, we find that the detailed balance condition is recovered through the coarse-graining procedure yielding Eq. (25) and that Eq. (10) can be understood as a result of the recovery of detailed balance with respect to the canonical distribution for the temperature Θ.

The second insight obtained from Eq. (25) is related to the extension of thermodynamics to a NESS. When we assume that $\tilde{V}(Y)$ takes a tanh-like form with amplitude Δ [that is, $\tilde{V}(\infty)-\tilde{V}(-\infty)=\Delta$], we have

$$\frac{Q_+ - Q_-}{Q_-} = -\frac{\Delta}{\Theta} + O\left(\left(\frac{\Delta}{\Theta}\right)^2\right) \tag{52}$$

with $Q_\pm \equiv \lim_{Y\to\pm\infty} Q(Y)$. Then, the chemical potential extended to NESSs can be defined using Eq. (52) in a similar way as in the case of a driven lattice gas [16]. Therefore, it may be possible to incorporate the idea of the effective temperature into a theoretical framework of thermodynamic functions extended to NESSs. A study with this aim treating a wide class of nonequilibrium systems, including many-body systems, is now in progress.

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[6] See, e.g., the Introduction of Ref. [9].
[7] This value represent the ratio of the mass $\sim 10^{-15}$ g to the resistance constant $\sim 10^{-6}$ g/sec.
[8] See, e.g., H. Risken, *The Fokker-Planck Equation* (Springer, Berlin, 1984).
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